Model reduction in a behavioral framework

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4 Dissipativity preserving model reduction by balanced truncation with error bounds

4.1 Introduction

In this chapter we revisit the problems of reduction by balancing of positive real and bounded real systems *from a behavioral point of view.* Positive real (or passive) and bounded real (or contractive) systems are special cases of systems that are dissipative on the negative half line, and whose number of input components is equal to the positive signature of the supply rate. In this paper we study the general problem of model reduction by balancing for strictly half line dissipative systems whose input cardinality equals the positive signature of the supply rate.

In the spirit of the work of Weiland [71], we introduce two crucial maps between the past and the future behavior. These maps assign to each past (future) trajectory its optimal future (past) in the sense of maximal extraction of available storage and minimal required supply. A difference with the set-up of Weiland is that in the context of strictly half line dissipative systems only the past behavior is an inner product space, while on the future behavior we only have an indefinite inner product. Yet, the system invariants can still be interpreted as singular values, in a slightly more general sense with a definite and an indefinite inner product.

In order to perform the actual balancing and balanced truncation, we represent our dissipative behavior by a normalized driving variable representation, more precisely, a driving variable representation whose transfer matrix from driving variable to manifest variable is inner with respect to the quadratic form of the supply rate. We show that, after balanced reduction, the property of *strict* dissipativity on the negative half line is preserved. To prove this, we need that asymptotic stability is preserved. In the present context this is a nontrivial matter, since the classical proof from [51] breaks down due to the indefinite structure of one of our Lyapunov equations.

Finally, we derive a frequency domain inequality for the error transfer matrix, i.e. the difference between the original and reduced order transfer matrix from driving variable to manifest variable. We show that for the special case of strictly bounded real systems this frequency domain inequality yields a new error bound for bounded real balancing.

4.2 Problem statement

As announced in the introduction section, the problem that we consider in this chapter is to approximate a given strictly half line dissipative behavior by a behavior whose McMillan degree is bounded from above by a given positive integer and that has the same input cardinality as the original behavior, such that the property of strict half line dissipativity is preserved. We restrict ourselves to behaviors with input cardinality equal to the positive signature of the supply rate. More precise, the problem can be formulated as follows:

Main Problem. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ and let $\Sigma = \Sigma^{\top} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_{-}$ and $\mathtt{m}(\mathfrak{B}) = \sigma_{+}(\Sigma)$. Let $\mathtt{k}$ be an integer such that $0 < \mathtt{k} < \mathtt{n}(\mathfrak{B})$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ such that

1. $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}_{-}$,
2. $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$,
3. $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$,
4. $\hat{\mathfrak{B}}$ is an approximation of $\mathfrak{B}$.

In effect, we will show that reduction by balancing leads to a behavior $\hat{\mathfrak{B}}$ satisfying properties 1,2 and 3. The question whether it is a reasonable approximation is studied afterwards, and amounts to finding reasonable error bounds.

Note that both in the case of strictly passive systems and strictly bounded real systems the condition $\mathtt{m}(\mathfrak{B}) = \sigma_{+}(\Sigma)$ is satisfied. Thus, our problem formulation captures both the problem of model reduction with preservation of strict passivity as well as the problem of model reduction with preservation of strict bounded realness.

4.3 Σ-characteristic values of system behaviors

In this section we introduce the notion of Σ-*characteristic values* of behaviors that are strictly Σ-dissipative on $\mathbb{R}_{-}$ and that have the property $\mathtt{m}(\mathfrak{B}) = \sigma_{+}(\Sigma)$.

Under the assumption that $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}_{-}$ the past behavior becomes an inner product space, with inner product given by the integral of the supply rate. The future behavior will then only be an indefinite inner product space. We will formulate a theorem that states that certain operators between past and future behavior allow singular value decompositions. This terminology should however be interpreted carefully, since the future behavior is not an inner product space. The "singular values" will form a set of invariants of the strictly Σ-dissipative behavior, and will be called the Σ-*characteristic values of* $\mathfrak{B}$.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let a supply rate be given by the nonsingular symmetric matrix $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume $\mathfrak{B}$ is strictly Σ-dissipative. In order to proceed, introduce the following notation:

$$\mathfrak{B}_- := \{w|_{\mathbb{R}_-} \mid w \in \mathfrak{B}\}, \quad \mathfrak{B}_+ := \{w|_{\mathbb{R}_+} \mid w \in \mathfrak{B}\}.$$

Furthermore, for a given past trajectory $w_- \in \mathfrak{B}_-$ define the set of all future trajectories w_+ whose concatenation at time zero with past trajectory w_- is in $\mathfrak{B}$ by

$$\mathfrak{B}_+(w_-) := \{w_+ \in \mathfrak{B}_+ \mid \ \exists\, w \in \mathfrak{B} \text{ s.t. } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\},$$

and for a given future trajectory $w_+ \in \mathfrak{B}_+$ define the set of all past trajectories w_- whose concatenation at time zero with future trajecory w_+ is in $\mathfrak{B}$ by

$$\mathfrak{B}_-(w_+) := \{w_- \in \mathfrak{B}_- \mid \ \exists\, w \in \mathfrak{B} \text{ s.t. } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\}.$$

For a given past trajectory $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we define the associated *available storage* by

$$V_{av}(w_-) := \sup\{-\int_0^\infty w_+^\top \Sigma w_+ dt \mid w_+ \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)\}, \qquad (4.1)$$

and for a given future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we define the associated *required supply* by

$$V_{req}(w_+) := \inf\{\int_{-\infty}^0 w_-^\top \Sigma w_- dt \mid w^- \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)\}. \qquad (4.2)$$

The available storage associated with past trajectory w_- is the maximal amount of supply that can be extracted from the system over all future trajectories $w_+ \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$. The required supply associated with future trajectory w_+ is the minimal amount of supply that has to be delivered to the system over all past trajectories $w_- \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$.

Due to Σ-dissipativity of $\mathfrak{B}$, the supremum and infimum above are finite for all w_- and w_+, respectively (see [79], [80], [81]). Also, by *strict* Σ-dissipativity, both the supremum and infimum are attained for all w_- and w_+. In particular, for given $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ there is a *unique* $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$ such that

$$V_{av}(w_-) = -\int_0^\infty w_+^{*\top} \Sigma w_+^* dt$$

and for given $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ there is a unique $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$ such that

$$V_{req}(w_+) = \int_{-\infty}^0 w_-^{*\top} \Sigma w_-^* dt.$$

By associating with any past trajectory $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$ we obtain a map

$$\Gamma_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+),\ \Gamma_-(w_-) = w_+^*,$$

and by associating with any future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$ we obtain a map

$$\Gamma_+ : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-),\ \Gamma_+(w_+) = w_-^*.$$

In the remainder of this section, assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$. This implies that the bilinear form

$$< w_1, w_2 >_{-,\Sigma} \quad := \quad \int_{-\infty}^0 w_1^\top \Sigma w_2 dt$$

defines an inner product on $\mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. On $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ consider the bilinear form

$$< w_1, w_2 >_{+,\Sigma} \quad := \quad -\int_0^\infty w_1^\top \Sigma w_2 dt.$$

This only defines an *indefinite* inner product on $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$. Now, in the sequel it will turn out that the maps Γ_- and Γ_+ are linear. We will denote by $\Gamma_-^* : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the *adjoint* of Γ_-, i.e. the (unique) linear map $\Gamma_-^* : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ that satisfies

$$< w_1, \Gamma_-(w_2) >_{+,\Sigma} \quad = \quad < \Gamma_-^*(w_1), w_2 >_{-,\Sigma}$$

for all $w_1 \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ and $w_2 \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. The existence and uniqueness of this adjoint can be easily proven, see e.g. [19], chapter 4. Likewise, $\Gamma_+^* : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ will denote the adjoint of Γ_+, i.e. the unique linear map that satisfies

$$< w_1, \Gamma_+(w_2) >_{-,\Sigma} \quad = \quad < \Gamma_+^*(w_1), w_2 >_{+,\Sigma}$$

for all $w_1 \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ and $w_2 \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$.

Recall from Section 3.2 the subbehavior of minimal dissipation $\mathfrak{B}^*$ and its antistable and stable parts $(\mathfrak{B}^*)_{\text{antistab}}$, $(\mathfrak{B}^*)_{\text{stab}}$. Denote $(\mathfrak{B}^*)^-_{\text{antistab}} := \{w|_{\mathbb{R}_-} \mid w \in (\mathfrak{B}^*)_{\text{antistab}}\}$ and $(\mathfrak{B}^*)^+_{\text{stab}} := \{w|_{\mathbb{R}_+} \mid w \in (\mathfrak{B}^*)_{\text{stab}}\}$. We now formulate a theorem stating that if $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then the maps Γ_- and Γ_+ allow *singular value decompositions* that, in a certain sense, are compatible. It should however be understood that, strictly speaking, the terminology singular value decomposition is not appropriate in the present context, since our maps do not act between genuine inner product spaces: only the past behavior is an inner product space, on the future behavior we have an indefinite inner product. The notion singular value should therefore be interpreted in a generalized sense:

Theorem 4.3.1. *Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. The maps Γ_- and Γ_+ are linear. The map $\Gamma_-^*\Gamma_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ has a finite-dimensional image, and is Hermitian and nonnegative. There exists positive real numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}} > 0$, where $\mathtt{n} = \mathtt{n}(\mathfrak{B})$, the McMillan degree of $\mathfrak{B}$, such that $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_{\mathtt{n}}^2 > 0$ are the nonzero eigenvalues of $\Gamma_-^*\Gamma_-$. There exists an orthonormal set $\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\} \subset \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$, and an orthonormal set $\{w_1^+, w_2^+, \ldots, w_{\mathtt{n}}^+\} \subset \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ such that*

$$\Gamma_- = \sum_{i=1}^{\mathtt{n}} \sigma_i < \cdot\, , w_i^- >_{-,\Sigma} w_i^+, \tag{4.3}$$

$$\Gamma_+ = \sum_{i=1}^{\mathtt{n}} \frac{1}{\sigma_i} < \cdot\, , w_i^+ >_{+,\Sigma} w_i^-. \tag{4.4}$$

Moreover,

$$(\mathfrak{B}^*)^-_{\text{antistab}} = \operatorname{span}\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\}, \tag{4.5}$$

and

$$(\mathfrak{B}^*)^+_{\text{stab}} = \operatorname{span}\{w_1^+, w_2^+, \ldots, w_{\mathtt{n}}^+\}. \tag{4.6}$$

Proof. A proof of this theorem will be given at the end of section 4.4. □

An important ingredient in the above theorem is nonnegativity of the map $\Gamma_-^*\Gamma_-$. Of course, in genuine inner product spaces this nonnegativity is a triviality. In the present context it is a statement that needs to be proven explicitely, and which follows from the fact that $\operatorname{im}(\Gamma_-)$ is a positive subspace for the indefinite future inner product. This follows from the nonnegativity of the available storage (which in turn follows from dissipativity on the negative half line and the assumption that $\mathtt{m}(\mathfrak{B}) = \sigma(\Sigma)$).

The positive real numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}} > 0$ will be called the Σ-*characteristic values of* $\mathfrak{B}$. As noted before, in a generalized sense these numbers are the singular values of the map Γ_-. In that sense, the pairs of functions (w_i^-, w_i^+) can be considered as Schmidt pairs of Γ_-.

Remark 4.3.2. In [71], an analogous theorem was proven in a slightly different context in which both past as well as future behavior were assumed to be inner product spaces. Using this genuine inner product structure, in [71] elementary least squares arguments were used to prove the theorem. In the present context, the proof given in [71] breaks down. In the sequel, we will present a proof of this theorem, starting from a minimal DV-representation of the behavior $\mathfrak{B}$.

4.4 State space characterizations and representations

In this section we review the characterizations of (strict) Σ-dissipativity in terms of the algebraic Riccati equation associated with a minimal DV-representation of the given behavior $\mathfrak{B}$. We explicitely compute representations of the linear maps (and their adjoints) that assign to each past (future) trajectory the unique state at time zero, and we characterize the extremal solutions of the Riccati equation in terms of these maps. We also compute the maps Γ_- and Γ_+ in terms of compositions of these maps. This will enable us then to give a proof of Theorem 4.3.1. Finally, we show that the Σ-characteristic values are the eigenvalues of the product of the inverse of the maximal solution and the minimal solution of the algebraic Riccati equation, and prove that $\mathfrak{B}$ admits a Σ-balanced minimal DV-representation. Much of the material in this section is an extension of results in [71] to the case that the future behavior is an indefinite inner product space.

Recall that in this chapter we deal with the class of supply rate Σ such that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. In order to proceed we recall Proposition 1.6.15 on the characterizations of Σ-dissipativity in terms of the algebraic Riccati equation associated with a minimal DV-representation:

Proposition 4.4.1. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ *with minimal DV-representation* $\mathfrak{B}_{DV}(A,B,C,D)$ *and let* $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *be nonsingular. Assume* $D^\top \Sigma D > 0$*. Then*

1. $\mathfrak{B}$ *is* Σ*-dissipative if and only if there exists a real symmetric solution* $K = K^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *of the algebraic Riccati equation (ARE)*

$$A^\top K + KA - C^\top \Sigma C + (KB - C^\top \Sigma D)(D^\top \Sigma D)^{-1}(B^\top K - D^\top \Sigma C) = 0.$$

$$(4.7)$$

If this is the case, then there exist real symmetric solutions K_- and K_+ such that every real symmetric solution K satisfies $K_- \leq K \leq K_+$.

2. *$\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if there exists a positive semidefinite solution $K = K^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ of the ARE (4.7).*

3. *If $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ then $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if all solutions of ARE (4.7) are positive definite, equivalently $K_- > 0$.*

We also restate Proposition 1.6.18 in a convenient way for the purposes of this chapter.

Proposition 4.4.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. If $\mathfrak{B}$ is strictly Σ-dissipative then $D^\top\Sigma D > 0$, and the minimal and maximal real symmetric solution K_- and K_+ of the ARE (4.7) satisfy $K_+ > K_-$. Furthermore, K_- and K_+ are stabilizing and anti-stabilizing, respectively, i.e., $\sigma(A_-) \subset \mathbb{C}_-$ and $\sigma(A_+) \subset \mathbb{C}_+$, where we denote*

$$A_+ := A + B(D^\top\Sigma D)^{-1}(B^\top K_+ - D^\top\Sigma C), \tag{4.8}$$

$$A_- := A + B(D^\top\Sigma D)^{-1}(B^\top K_- - D^\top\Sigma C). \tag{4.9}$$

Finally, the following statements are equivalent*:*

1. *$\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$,*

2. *$D^\top\Sigma D > 0$ and the maximal solution K_+ of the ARE (4.7) is positive definite and anti-stabilizing, i.e., $\sigma(A_+) \subset \mathbb{C}_+$.*

Proof. A proof of the claim that $K_+ > K_-$ is contained in the proof of Theorem 5.7 in [80]. Proofs of the statement $D^\top\Sigma D > 0$, and the equivalence of statements 1. and 2. can be given similar as to the proof of Theorem 5.3.4 in [41]. There, it was also shown that strict Σ-dissipativity implies that the Hamiltonian matrix associated with our ARE has no imaginary eigenvalues. This implies that K_- and K_+ must be stabilizing and anti-stabilizing, respectively. We omit the details. □

We will now study the maps Γ_- and Γ_+ in terms of DV-representations of the given behavior $\mathfrak{B}$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$. Let $\mathtt{n} = \mathtt{n}(\mathfrak{B})$ be the McMillan degree of $\mathfrak{B}$. By minimality, for any $w \in \mathfrak{B}$ there is a unique state trajectory x such that $\frac{d}{dt}x = Ax + Bv$, $w = Cx + Dv$. For any given $x_0 \in \mathbb{R}^{\mathtt{n}}$, the state space, let $\mathfrak{B}(x_0)$ denote the set of all $w \in \mathfrak{B}$ such that the corresponding state trajectory x satisfies $x(0) = x_0$. Thus, for every $w \in \mathfrak{B}$ there is a unique $x_0 \in \mathbb{R}^{\mathtt{n}}$ such that $w \in \mathfrak{B}(x_0)$. Moreover (see [71]), there exists linear surjective maps $R_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathbb{R}^{\mathtt{n}}$ and $R_+ : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathbb{R}^{\mathtt{n}}$ such that for all $x_0 \in \mathbb{R}^{\mathtt{n}}$ we have

$$w \in \mathfrak{B}(x_0) \quad \Leftrightarrow \quad \{R_-(w_-) = x_0 \text{ and } R_+(w_+) = x_0\},$$

where $w_- := w|_{\mathbb{R}_-}$ and $w_+ := w|_{\mathbb{R}_+}$. In the sequel we will explicitely compute representations of the maps R_- and R_+, and their adjoints R_-^* and R_+^* in terms of the systems matrices A,B,C and D. On $\mathbb{R}^{\mathtt{n}}$ we take the standard Euclidean inner product. Note that R_+^* denotes the generalized adjoint with respect to the indefinite inner product on $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$.

It is well known (see [66]) that the extremal solutions of the Riccati equation (4.7) are associated with the available storage and required supply.

Proposition 4.4.3. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$. Assume that $D^\top \Sigma D > 0$. Assume $\mathfrak{B}$ is Σ-dissipative and let K_- and K_+ be the minimal and maximal real symmetric solution of the ARE (4.7). Then for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we have $V_{av}(w_-) = x_0^\top K_- x_0$, where $x_0 := R_-(w_-)$. Also, for any $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we have $V_{req}(w_+) = x_0^\top K_+ x_0$, where $x_0 := R_+(w_+)$.*

If $\mathfrak{B}$ is strictly Σ-dissipative then K_- and K_+ satisfy $\sigma(A_+) \subset \mathbb{C}_+$ and $\sigma(A_-) \subset \mathbb{C}_-$ (see Theorem 4.4.1). Introduce the following notation:

$$C_+ := C + D(D^\top \Sigma D)^{-1}(B^\top K_+ - D^\top \Sigma C), \tag{4.10}$$

$$C_- := C + D(D^\top \Sigma D)^{-1}(B^\top K_- - D^\top \Sigma C). \tag{4.11}$$

The following is also well-known (see [73]):

Proposition 4.4.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$. Assume $\mathfrak{B}$ is strictly Σ-dissipative. Then for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory w_+^* is given by $w_+^*(t) = C_- e^{A_- t} x_0$, where $x_0 := R_-(w_-)$. Also, for any $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory w_-^* is given by $w_-^*(t) = C_+ e^{A_+ t} x_0$, where $x_0 := R_+(w_+)$.*

In the remainder of this section we will assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. In this case, in addition we have $0 < K_- < K_+$. The next theorem is the main result of this section. It computes representations of R_- and R_+ and their adjoints R_-^* and R_+^*, and shows that K_- and K_+ can be expressed in terms of compositions of these maps.

Theorem 4.4.5. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ and $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we have*

$$R_-(w_-) = \int_{-\infty}^{0} e^{-(A_+ - K_+^{-1}C_+^\top \Sigma C_+)s} K_+^{-1} C_+^\top \Sigma w_-(s) ds, \tag{4.12}$$

$$R_+(w_+) = -\int_{0}^{\infty} e^{-(A_- - K_-^{-1}C_-^\top \Sigma C_-)s} K_-^{-1} C_-^\top \Sigma w_+(s) ds. \tag{4.13}$$

Furthermore, for any $x_0 \in \mathbb{R}^{\mathtt{n}}$ we have

$$R_-^*(x_0) = C_+ K_+^{-1} e^{-(A_+ - K_+^{-1}C_+^\top \Sigma C_+)^\top t} x_0 \tag{4.14}$$

and

$$R_+^*(x_0) = C_- K_-^{-1} e^{-(A_- - K_-^{-1}C_-^\top \Sigma C_-)^\top t} x_0. \tag{4.15}$$

Finally, $K_+ = (R_- R_-^*)^{-1}$ and $K_- = (R_+ R_+^*)^{-1}$.

Proof. It is easily verified from the ARE (4.7) that $A_+^\top K_+ + K_+ A_+ - C_+^\top \Sigma C_+ = 0$ and $A_-^\top K_- + K_- A_- - C_-^\top \Sigma C_- = 0$. This yields

$$K_+^{-1} A_+^\top + A_+ K_+^{-1} - K_+^{-1} C_+^\top \Sigma C_+ K_+^{-1} = 0 \tag{4.16}$$

and

$$K_-^{-1} A_-^\top + A_- K_-^{-1} - K_-^{-1} C_-^\top \Sigma C_- K_-^{-1} = 0. \tag{4.17}$$

We claim that $A_+ - K_+^{-1} C_+^\top \Sigma C_+$ is similar to $-A_+^\top$. Indeed, from (4.16) we have $-K_+^{-1} A_+^\top = (A_+ - K_+^{-1} C_+^\top \Sigma C_+) K_+^{-1}$ so

$$K_+ (A_+ - K_+^{-1} C_+^\top \Sigma C_+) K_+^{-1} = -A_+^\top. \tag{4.18}$$

As a consequence, $\sigma(A_+ - P_+^{-1} C_+^\top \Sigma C_+) \subset \mathbb{C}_-$. In the same way we show that $\sigma(A_- - K_-^{-1} C_-^\top \Sigma C_-) \subset \mathbb{C}_+$. Also, note that $C_+^\top \Sigma D = K_+ B$ and $C_-^\top \Sigma D = K_- B$.

It is easily seen that $\mathfrak{B}_{DV}(A_+,B,C_+,D)$ and $B_{DV}(A_-,B,C_-,D)$ both provide a minimal driving variable representation of $\mathfrak{B}$. We will now prove (4.12). Let $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. There exist x,v such that $\dot{x} = A_+x + Bv$ and $w_- = C_+x + Dv$. This yields $C_+^\top \Sigma w_- = C_+^\top \Sigma C_+ x + C_+^\top \Sigma D v = C_+^\top \Sigma C_+ x + K_+ Bv$. Consequently,

$$Bv = K_+^{-1} C_+^\top \Sigma w_- - K_+^{-1} C_+^\top \Sigma C_+ x.$$

Thus the state trajectory x corresponding to w_- satisfies $\dot{x} = (A_+ - K_+^{-1} C_+^\top \Sigma C_+)x + K_+^{-1} C_+^\top \Sigma w_-$. Since $A_+ - K_+^{-1} C_+^\top \Sigma C_+$ is stable, this implies that $x(0) = R_-(w_-)$ is given by (4.12). In the same way, working with the driving variable representation $B_{DV}(A_-,B,C_-,D)$, we can prove (4.13).

We will now prove (4.14). Let $x_0 \in \mathbb{R}^n$ and let $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. Let (x,y) denote the Euclidean inner product on $\mathbb{R}^n$. We have

$$(x_0, R_-(w_-)) = \int_{-\infty}^0 x_0^\top e^{-(A_+ - K_+^{-1} C_+^\top \Sigma C_+)s} K_+^{-1} C_+^\top \Sigma w_-(s) ds.$$

The latter should be equal to $< R_-^*(x_0), w_- >_{-,\Sigma}$ so $R_-(x_0)$ must be given by (4.14). In the same way we can prove (4.15).

Finally, for any $x_0 \in \mathbb{R}^n$ we have

$$(R_- R_-^*)(x_0) = \int_{-\infty}^0 e^{-(A_+ - K_+^{-1} C_+^\top \Sigma C_+)s} K_+^{-1} C_+^\top \Sigma C_+ K_+^{-1} e^{-(A_+ - K_+^{-1} C_+^\top \Sigma C_+)^\top s} ds x_0.$$

It is easily verified that the integral on the right is equal to the unique solution X of the equation

$$(A_+ - K_+^{-1} C_+^\top \Sigma C_+)X + X(A_+ - K_+^{-1} C_+^\top \Sigma C_+)^\top + K_+^{-1} C_+^\top \Sigma C_+ K_+^{-1} = 0,$$

which yields $X = K_+^{-1}$ by virtue of equation (4.16). We conclude that $R_- R_-^* = K_+^{-1}$. In the same way we can prove that $R_+ R_+^* = K_-^{-1}$. □

Remark 4.4.6. If the past and the future behavior are inner product spaces a result analogous to $K_+ = (R_- R_-^*)^{-1}$ and $K_- = (R_+ R_+^*)^{-1}$ was proven in [71] using a general least squares argument, without computing explicit representations of R_-, R_-^*, R_+ and R_+^*.

Corollary 4.4.7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then we have $\Gamma_- = R_+^*(R_+R_+^*)^{-1}R_-$ and $\Gamma_+ = R_-^*(R_-R_-^*)^{-1}R_+$.*

Proof. By Proposition 4.4.4 we know that for $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we have $\Gamma_-(w_-) = C_- e^{A_- t} x_0$, with $x_0 = R_-(w_-)$. By (4.17) we have $A_- = -K_-^{-1}(A_- - K_-^{-1}C_-^\top \Sigma C_-)^\top K_-$ so

$$\begin{aligned}\Gamma_-(w_-) &= C_- e^{-K_-^{-1}(A_- - K_-^{-1}C_-\Sigma C_-)^\top K_- t} x_0 \\ &= C_- K_-^{-1} e^{-(A_- - K_-^{-1}C_-\Sigma C_-)^\top t} K_- x_0.\end{aligned}$$

By Theorem 4.4.5 this is equal to $(R_+^*(R_+R_+^*)^{-1}R_-)(w_-)$. In the same way a proof for Γ_+ can be given. □

We now prove that the numbers $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_{\mathtt{n}}^2 > 0$ are in fact the eigenvalues of $K_+^{-1}K_-$, with $0 < K_- < K_+$ the extremal solutions of the ARE (4.7), for any minimal DV-representation of $\mathfrak{B}$.

Theorem 4.4.8. *Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}} > 0$ be the Σ-characteristic values of $\mathfrak{B}$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV-representation of $\mathfrak{B}$ with $0 < K_- < K_+$ the extremal solutions of the ARE (4.7). Then $\{\sigma_1^2,\sigma_2^2,\ldots,\sigma_{\mathtt{n}}^2\} = \sigma(K_+^{-1}K_-)$. Furthermore $0 < \sigma_i < 1$ for all i.*

Proof. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal driving variable representation of $\mathfrak{B}$. For $i = 1,2\ldots,\mathtt{n}$ there exist $w_i^- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ such that $(\Gamma_-^*\Gamma_-)(w_i^-) = \sigma_i^2 w_i^-$. This implies $(R_-R_-^*(R_+R_+^*)^{-1}R_-)(w_i^-) = \sigma_i^2 R_-(w_i^-)$. Now, $R_-(w_i^-) \neq 0$, for otherwise we must have $\sigma_i = 0$. Thus $\sigma_i^2 \in \sigma(K_+^{-1}K_-)$. Conversely, let λ and $x \neq 0$ be such that $K_+^{-1}K_- x = \lambda x$. Then $\Gamma_-^*\Gamma_- R_-^*(R_+R_+^*)^{-1}x = \lambda R_-^*(R_+R_+^*)^{-1}x$. By surjectivity of R_- we have $R_-^*(R_+R_+^*)^{-1}x \neq 0$. Thus $\lambda = \sigma_i^2$ or some i.

We finally prove $\sigma_i < 1$ for all i. From $0 < K_- < K_+$, we obtain $K_+^{-\frac{1}{2}} K_- K_+^{-\frac{1}{2}} < I$. The claim follows from the fact that the eigenvalues of $K_+^{-\frac{1}{2}} K_- K_+^{-\frac{1}{2}}$ and $K_+^{-1}K_-$ coincide. □

With the help of some results in this section, we are now in a position to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. The claim that Γ_- and Γ_+ are linear follows immediately from Corollary 4.4.7. Next, note that $\Gamma_-^*\Gamma_- = R_-^*(R_+R_+^*)^{-1}R_-$. Since R_- is surjective and $R_+R_+^{-1}$ maps $\mathbb{R}^{\mathtt{n}}$ onto itself, $\Gamma_-^*\Gamma_-$ has an $\mathtt{n}$-dimensional image and has therefore $\mathtt{n}$ nonzero eigenvalues (see [83]). It is easily verified that $\Gamma_-^*\Gamma_-$ is Hermitian. The fact that it is nonnegative can be proved using the fact that the available storage is nonnegative: for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we have $< w_-,(\Gamma_-^*\Gamma_-)(w_-) >_{-,\Sigma} = < \Gamma_-(w_-),\Gamma_-(w_-) >_{+,\Sigma} =$

$V_{av}(w_-) = x_0^\top K_- x_0 \geq 0$, where $x_0 = R_-(w_-)$. From this it follows that $\Gamma_-^*\Gamma_-$ has $\mathtt{n}$ positive eigenvalues, say $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_{\mathtt{n}}^2$. By [83], theorem 8.15 there exists an orthonormal set $\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\}$ of eigenvectors, $(\Gamma_-^*\Gamma_-)(w_i^-) = \sigma_i^2 w_i^-$. Now define $w_i^+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ by $w_i^+ := \frac{1}{\sigma_i}\Gamma_-(w_i^-)$. We prove that $\{w_1^+, w_2^+, \ldots, w_{\mathtt{n}}^+\}$ is an orthonormal subset of $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ (in the *indefinite* inner product). Indeed, $< w_i^+, w_j^+ >_{+,\Sigma} = < \frac{1}{\sigma_i}\Gamma_-(w_i^-), \frac{1}{\sigma_j}\Gamma_-(w_j^-) >_{+,\Sigma} =< \frac{1}{\sigma_i}w_i^-, \frac{1}{\sigma_j}(\Gamma_-^*\Gamma_-)(w_j^-) >_{-,\Sigma} = < \frac{1}{\sigma_i}w_i^-, \sigma_j w_j^- >_{-,\Sigma} = \delta_{ij}$. Obviously we have $\Gamma_-(w_i^-) = \sigma_i w_i^+$. We now prove (4.3). Since the image of Γ_- is $\mathtt{n}$-dimensional, the set $\{w_1^+, w_2^+, \ldots, w_{\mathtt{n}}^+\}$ (being linearly independent) forms a basis of this image. Let $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. Then there exist μ_i such that $\Gamma_-(w_-) = \sum_{i=1}^{\mathtt{n}} \mu_i w_i^+$. We compute the μ_i as follows: $\mu_i =< \Gamma_-(w_-), w_i^+ >_{+,\Sigma} = < \Gamma_-(w_-), \frac{1}{\sigma_i}\Gamma_-(w_i^-) >_{+,\Sigma} = < w_-, \frac{1}{\sigma_i}(\Gamma_-^*\Gamma_-)(w_i^-) >_{-,\Sigma} = < w_-, \frac{1}{\sigma_i}\sigma_i^2 w_i^- >_{-,\Sigma} = \sigma_i < w_-, w_i^- >_{-,\Sigma}$. This proves (4.3).

Next, we prove (4.4). We first show that $\Gamma_+(w_i^+) = \frac{1}{\sigma_i}w_i^-$. By definition, $w_i^+ = \frac{1}{\sigma_i}\Gamma_-(w_i^-)$, so using Corollary 4.4.7 we have $\Gamma_+(w_i^+) = \frac{1}{\sigma_i}(\Gamma_+\Gamma_-)(w_i^-)$. Also, $\Gamma_+\Gamma_- = R_-^*(R_-R_-^*)^{-1}R_-$. Since w_i^- is an eigenvector of $\Gamma_-^*\Gamma_-$, we have $w_i^- \in \operatorname{im}(\Gamma_-^*) \subset \operatorname{im}(R_-^*)$. Hence there exists v_i such that $w_i^- = R_-^* v_i$. This implies that $\Gamma_+(w_i^+) = \frac{1}{\sigma_i}R_-^*(v_i) = \frac{1}{\sigma_i}w_i^-$. Finally, since Γ_+ has an $\mathtt{n}$-dimensional image with basis $\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\}$, the remainder of the proof can be given along the lines of the corresponding result for Γ_-.

Next, we prove that $(\mathfrak{B}^*)^-_{\text{antistab}} = \text{span}\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\}$. Recall from Lemma 3.4.2 that

$$(\mathfrak{B}^*)_{\text{antistab}} = \text{span}\{((C - D(D^\top\Sigma D)^{-1}D^\top\Sigma C)X_1 + D(D^\top\Sigma D)^{-1}B^\top Y_1)e^{Mt}\},$$

where $X_1, Y_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ are such that the columns of $\operatorname{col}(X_1, Y_1)$ form a basis of $\mathcal{X}_+(H)$, $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is the matrix of $H\,|_{\mathcal{X}_+(H)}$ with respect to this basis, and H is the Hamiltonian matrix given by

$$H := \begin{bmatrix} A - B(D^\top\Sigma D)^{-1}D^\top\Sigma C & B(D^\top\Sigma D)^{-1}B^\top \\ C^\top\Sigma C - C^\top\Sigma D(D^\top\Sigma D)^{-1}D^\top\Sigma C & -(A - B(D^\top\Sigma D)^{-1}D^\top\Sigma C)^\top \end{bmatrix}.$$

In terms of $A_+ = A + B(D^\top\Sigma D)^{-1}(B^\top K_+ - D^\top\Sigma C)$ and $C_+ = C + D(D^\top\Sigma D)^{-1}(B^\top K_+ - D^\top\Sigma C)$, the Hamiltonian matrix H and the subbehavior $(\mathfrak{B}^*)_{\text{antistab}}$ can be written as:

$$H = \begin{bmatrix} A_+ - B(D^\top\Sigma D)^{-1}B^\top K_+ & B(D^\top\Sigma D)^{-1}B^\top \\ C_+^\top\Sigma C_+ - K_+B(D^\top\Sigma D)^{-1}B^\top K_+ & -(A_+ - B(D^\top\Sigma D)^{-1}B^\top K_+)^\top \end{bmatrix},$$

$(\mathfrak{B}^*)_{\text{antistab}} = \text{span}\{((C_+ - D(D^\top\Sigma D)^{-1}B^\top P_+)X_1 + D(D^\top\Sigma D)^{-1}B^\top Y_1)e^{Mt}\}$.

It is easy to see that $X_1 = K_+^{-1}$, $Y_1 = I$ and $M = K_+A_+K_+^{-1}$ satisfy the equation: $H\ \text{col}(X_1,Y_1) = \text{col}(X_1,Y_1)\ M$. Therefore

$$\begin{aligned}(\mathfrak{B}^*)_{\text{antistab}} &= \text{span}\{((C_+ - D(D^\top\Sigma D)^{-1}B^\top K_+)K_+^{-1} \\ &\qquad\qquad + D(D^\top\Sigma D)^{-1}B^\top I)e^{K_+A_+K_+^{-1}t}\} \\ &= \text{span}\{C_+K_+^{-1}e^{K_+A_+K_+^{-1}t}\}.\end{aligned}$$

On the other hand, let $(\sigma_i^2,x_i), i = 1\ldots\mathtt{n}$ be the eigenvalues and the eigenvectors of $K_+^{-1}K_-$, i.e. $K_+^{-1}K_-x_i = \sigma_i^2x_i$. By the proof of Theorem 4.4.8, we get

$$\begin{aligned}w_i^- &= R_-^*(R_+R_-^*)^{-1}x_i = R_-^*K_-x_i = C_+K_+^{-1}e^{-(A_+-K_+^{-1}C_+^\top\Sigma C_+)^\top t}K_-x_i \\ &= C_+K_+^{-1}e^{K_+^\top A_+K_+^{-\top}t}K_-x_i.\end{aligned}$$

This is implies that $\text{span}\{w_1^-,w_2^-,\ldots,w_{\mathtt{n}}^-\} \subseteq (\mathfrak{B}^*)^-_{\text{antistab}}$. However, since both $\text{span}\{w_1^-,w_2^-,\ldots,w_{\mathtt{n}}^-\}$ and $(\mathfrak{B}^*)^-_{\text{antistab}}$ have the same dimension $\mathtt{n}$, we conclude that

$$(\mathfrak{B}^*)^-_{\text{antistab}} = \text{span}\{w_1^-,w_2^-,\ldots,w_{\mathtt{n}}^-\},$$

as was to be proved.

Finally, a proof of $(\mathfrak{B}^*)^-_{\text{stab}} = \text{span}\{w_1^+,w_2^+,\ldots,w_{\mathtt{n}}^+\}$ can be given similarly. □

After any coordinate transformation $\hat{x} = Tx$ in the state space of the DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$, the maps R_- and R_+ transform to TR_- and TR_+. Thus $R_-R_-^*$ transforms to $TR_-R_-^*T^\top$ and $R_+R_+^*$ to $TR_+R_+^*T^\top$. This implies that K_+^{-1} transforms to $TK_+^{-1}T^\top$ and K_- to $T^{-\top}K_-T^{-1}$. It is well known, see e.g. [85], that there exists a coordinate transformation T such that K_- and K_+^{-1} are equal and diagonal. Since the set of eigenvalues of $K_+^{-1}K_-$ is $\{\sigma_1^2,\sigma_2^2,\ldots,\sigma_{\mathtt{n}}^2\}$ this diagonal matrix must be equal to the diagonal matrix $\Pi := \text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{\mathtt{n}})$. Thus we obtain:

Corollary 4.4.9. *Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}} > 0$ be the Σ-characteristic values of $\mathfrak{B}$. There exists a minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ such that the corresponding extremal solutions K_- and K_+ of the ARE (4.7) satisfy $K_- = K_+^{-1} = \text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{\mathtt{n}})$.*

A minimal DV-representation of $\mathfrak{B}$ such that $K_- = K_+^{-1} = \text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{\mathtt{n}})$ is called a *Σ-balanced DV-representation of* $\mathfrak{B}$.

4.5 Σ-normalized Σ-balanced DV-representations

In this section we show that if $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then it has a Σ-balanced minimal DV-representation that is, in addition, Σ-normalized in the following sense.

The idea of Σ-normalization originates from the concept of *normalized coprime factorization*, see e.g. [85]. In the following lemma we prove that strictly Σ-dissipative behaviors allow Σ-normalized DV-representations.

Lemma 4.5.1. *Assume that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ is strictly Σ-dissipative. Then there exists a minimal driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ such that A is asymptotically stable and $G^\top(-\xi)\Sigma G(\xi) = I$, where $G(\xi) := C(\xi I - A)^{-1}B + D$.*

Proof. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal driving variable representation of $\mathfrak{B}$. Let K_- be the smallest solution of the ARE (4.7). By Theorem 4.4.1, K_- is a stabilizing solution, so A_- given by (4.9) is asymptotically stable. Let C_- be given by (4.11). Since $D^\top \Sigma D > 0$ there exists a nonsingular R such that $D^\top\Sigma D = R^\top R$. Clearly,

$$\begin{aligned} \dot{x} &= A_- x + BR^{-1}\bar{v}, \\ w &= C_- x + DR^{-1}\bar{v} \end{aligned}$$

also generates a minimal representation $\mathfrak{B}_{DV}(A_-,BR^{-1},C_-,DR^{-1})$ of $\mathfrak{B}$. It is not difficult to check that $\bar{G}^\top(-\xi)\Sigma\bar{G}(\xi) = I$, with $\bar{G}(\xi) = C_-(\xi I - A_-)^{-1}BR^{-1} + DR^{-1}$. Finally, rename $(A_-,BR^{-1},C_-,DR^{-1})$ to (A,B,C,D). □

A driving variable representation satisfying the two conditions of Lemma 4.5.1 is called a Σ-*normalized driving variable representation* of $\mathfrak{B}$. Lemma 4.5.1 says that if a controllable behavior $\mathfrak{B}$ is strictly Σ-dissipative, then it admits a Σ-normalized minimal driving variable representation. Σ-normalized representations have some nice properties. This is elaborated in the following lemma.

Lemma 4.5.2. *Let $G(\xi)$ be proper rational and let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization. Assume $\sigma(A) \subset \mathbb{C}^-$. Let M be the unique symmetric solution of $A^\top M + MA - C^\top\Sigma C = 0$. If $B^\top M - D^\top \Sigma C = 0$ and $D^\top\Sigma D = I$, then $G^\top(-\xi)\Sigma G(\xi) = I$. If, in addition, (A,B) is controllable and (C,A) is observable, then $B^\top M - D^\top\Sigma C = 0$ and $D^\top \Sigma D = I$* if and only if *$G^\top(-\xi)\Sigma G(\xi) = I$.*

Proof. A proof can be given by slightly adapting the proof of Corollary 12.9 in [85] by including Σ in the proof. □

The above lemma suggest that for Σ-normalized representations the controllability and generalized observability (or Σ-observability) Gramians are related to the maximal and minimal solutions of the ARE. Indeed, we have:

Lemma 4.5.3. *Assume that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ is strictly Σ-dissipative. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-normalized driving variable representation of $\mathfrak{B}$. Let K_- and K_+ be the minimal and maximal real symmetric solutions of the ARE (4.7). Let M be the unique solution of $A^\top M + MA - C^\top \Sigma C = 0$ (called the* Σ-observability Gramian*), and let W be the unique solution of $AW + WA^\top + BB^\top = 0$ (the controllability Gramian). Then we have*

1. *$M = K_-$,*
2. *$W = (K_+ - K_-)^{-1}$.*

Proof. 1) It follows from Lemma 4.5.2 that M satisfies the ARE. Since A is asymptotically stable, M is also a stabilizing solution. We conclude that $M = K_-$.

2) Using $A^\top K_- + K_- A = C^\top \Sigma C$, $B^\top K_- = D^\top \Sigma C$ and $D^\top \Sigma D = I$ we get

$$\begin{aligned}
&(K_+ - K_-)A + A^\top (K_+ - K_-) + (K_+ - K_-)BB^\top (K_+ - K_-) \\
&\quad = K_+A + A^\top K_+ - K_-A - A^\top K_- \\
&\qquad +(K_+B - K_-B)(B^\top K_+ - B^\top K_-) \\
&\quad = K_+A + A^\top K_+ - C^\top \Sigma C \\
&\qquad +(K_+B - C^\top \Sigma D)(D^\top \Sigma D)^{-1}(B^\top K_+ - D^\top \Sigma C) \\
&\quad = 0, \quad \text{(since } K_+ \text{ is solution of ARE (4.7))}.
\end{aligned}$$

Therefore,

$$A(K_+ - K_-)^{-1} + (K_+ - K_-)^{-1}A^\top + BB^\top = 0,$$

so $(K_+ - K_-)^{-1} = W$, the unique solution of the Lyapunov equation $AW + WA^\top + BB^\top = 0$. □

If we start with a Σ-normalized driving variable representation, then transforming into Σ-balanced coordinates results in a Σ-normalized driving variable representation as well. This follows immediately from the fact that the transfer matrix associated with the DV-representation does not change under coordinate transformation. Thus we obtain:

Corollary 4.5.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be strictly Σ-dissipative on $\mathbb{R}_-$ and assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then there exists a Σ-normalized, Σ-balanced minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$.*

For a Σ-normalized, Σ-balanced minimal DV-representations of $\mathfrak{B}$, the Σ-observability Gramian and controllability Gramian are diagonal matrices, and their diagonal elements can be expressed in terms of the Σ-characteristic values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}}$ of $\mathfrak{B}$:

Lemma 4.5.5. *Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-normalized, Σ-balanced minimal DV-representation of $\mathfrak{B}$. Then the Σ-observability Gramian M and controllability Gramian W are given by*

1. $M = \Pi = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\mathtt{n}})$,
2. $W = (\Pi^{-1} - \Pi)^{-1} = \mathrm{diag}(\frac{\sigma_1}{1-\sigma_1^2}, \ldots, \frac{\sigma_{\mathtt{n}}}{1-\sigma_{\mathtt{n}}^2})$.

Proof. This is an immediate consequence of Lemma 4.5.3. □

4.6 Reduction by balanced truncation

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ be strictly Σ-dissipative on $\mathbb{R}_-$ and assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal Σ-normalized and Σ-balanced DV-representation of $\mathfrak{B}$. Define $G(\xi) = C(\xi I - A)^{-1}B + D$. We have $K_+^{-1} = K_- = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\mathtt{n}}) = \Pi$.

Pick $\mathtt{k} < \mathtt{n}$ such that $\sigma_{\mathtt{k}} > \sigma_{\mathtt{k}+1}$. Our aim is to compute a reduced order approximation $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ of $\mathfrak{B}$ with $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$, such that its McMillan degree $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$, and $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}_-$. In order to do this, perform the following steps:

Step 1. Partition Π into

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \tag{4.19}$$

where $\Pi_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\mathtt{k}})$, $\Pi_2 = \mathrm{diag}(\sigma_{\mathtt{k}+1}, \ldots, \sigma_{\mathtt{n}})$.

Step 2. Partition A, B and C comformably with the partitioning of Π:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{4.20}$$

and define the truncated system $\mathfrak{B}_{\mathrm{trunc}}$ by $\mathfrak{B}_{\mathrm{trunc}} := B_{DV}(A_{11}, B_1, C_1, D)$.

Step 3. Define our reduced order approximation as the controllable part of $\mathfrak{B}_{\text{trunc}}$:

$$\hat{\mathfrak{B}} := (\mathfrak{B}_{\text{trunc}})_{\text{cont}}. \tag{4.21}$$

A DV-representation of $\hat{\mathfrak{B}}$ can obtained by performing a Kalman controllability decomposition (see [**?**], proposition 22):

$$T^{-1}A_{11}T = \begin{bmatrix} \hat{A} & * \\ 0 & * \end{bmatrix}, T^{-1}B_1 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, C_1T = \begin{bmatrix} \hat{C} & * \end{bmatrix}, D = \hat{D}. \tag{4.22}$$

We then have $\hat{\mathfrak{B}} = \mathfrak{B}_{DV}(\hat{A},\hat{B},\hat{C},\hat{D})_{\text{ext}}$.

A major difference between our work and [71] is that in general the pair (A_{11},B_1) need not be controllable, and A_{11} need not be asymptotically stable. The proof given in [71], theorem 5.15 (which basically uses the classical result from [51]), breaks down due to the fact that in our second Lyapunov equation, $A^\top M+MA-C^\top\Sigma C = 0$, the constant term $-C^\top\Sigma C$ is indefinite. Since our reduced order system $\hat{\mathfrak{B}}$ should be controllable (which is a necessary condition for dissipativity), we are forced to take the controllable part of the truncated behavior $\mathfrak{B}_{\text{trunc}}$. Proving asymptotic stability of the resulting $\hat{A}$ remains a nontrivial matter. We now show however that the reduced order behavior $\hat{\mathfrak{B}}$ obtained in this way satisfies the required properties.

Theorem 4.6.1. *Let $\hat{\mathfrak{B}}$ be defined by (4.21). Then $\hat{\mathfrak{B}}$ is controllable. Furthermore, $\sigma(\hat{A}) \subset \mathbb{C}^-$, and $\mathfrak{B}_{DV}(\hat{A},\hat{B},\hat{C},\hat{D})$ is a Σ-normalized DV-representation of $\hat{\mathfrak{B}}$. Finally, $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$, $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$, and $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$.*

Proof. 1.) Clearly, $\hat{\mathfrak{B}}$ is controllable by Step 3.

2.) To prove that $\hat{A}$ is asymptotically stable is non-trivial. Since $\mathfrak{B} = \mathfrak{B}_{DV}(A,B,C,D)$ is Σ-normalized and Σ-balanced, $\Pi = \text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{\mathtt{n}})$ satisfies

$$A^\top\Pi + \Pi A - C^\top\Sigma C = 0,$$

$$B^\top\Pi - D^\top\Sigma C = 0,$$

$$D^\top\Sigma D = I,$$

$$A^\top\Pi^{-1} + \Pi^{-1}A - C^\top\Sigma C + (\Pi^{-1}B - C^\top\Sigma D)(B^\top\Pi^{-1} - D^\top\Sigma C) = 0.$$

Partition Π as in (4.19), and partition A,B and C as in (4.20). Then $\Pi_1 = \text{diag}(\sigma_1,\ldots,\sigma_{\mathtt{k}})$ satisfies

$$A_{11}^\top \Pi_1 + \Pi_1 A_{11} - C_1^\top \Sigma C_1 = 0,$$

$$B_1^\top \Pi_1 - D^\top \Sigma C_1 = 0,$$

$$A_{11}^\top \Pi_1^{-1} + \Pi_1^{-1} A_{11} - C_1^\top \Sigma C_1 + (\Pi_1^{-1} B_1 - C_1^\top \Sigma D)(B_1^\top \Pi_1^{-1} - D^\top \Sigma C_1) = 0.$$

Now consider the Kaman decomposition (4.22). Conformably partition

$$T^\top \Pi_1 T = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{pmatrix}, \quad T^\top \Pi_1^{-1} T = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{pmatrix}.$$

It is then easily seen that

$$\hat{A}^\top P_{11} + P_{11}\hat{A} - \hat{C}^\top \Sigma \hat{C} = 0, \tag{4.23}$$

$$\hat{B}^\top P_{11} - \hat{D}^\top \Sigma \hat{C} = 0, \tag{4.24}$$

$$\hat{A}^\top Q_{11} + Q_{11}\hat{A} - \hat{C}^\top \Sigma \hat{C} + (Q_{11}\hat{B} - \hat{C}^\top \Sigma \hat{D})(\hat{B}^\top Q_{11} - \hat{D}^\top \Sigma \hat{C}) = 0.$$

Since $\Pi_1^{-1} > \Pi_1$, we have $Q_{11} > P_{11}$. Define $R_{11} := (Q_{11} - P_{11})^{-1}$. Then $R_{11} > 0$ and similarly to the proof of Lemma 4.5.3 we obtain

$$\hat{A} R_{11} + R_{11}\hat{A}^\top + \hat{B}\hat{B}^\top = 0. \tag{4.25}$$

Since $R_{11} > 0$ and the pair $(\hat{A},\hat{B})$ is controllable, it follows from (4.25) that $\sigma(\hat{A}) \subset \mathbb{C}^-$.

3.) The fact that the DV-representation is Σ-normalized follows from Lemma 4.5.2 using (4.23), (4.24) and the fact that $\hat{D}^\top \Sigma \hat{D} = I$.

4.) We now prove that $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$. The reduced order system $\mathfrak{B}_{DV}(\hat{A},\hat{B},\hat{C},\hat{D})$ has $\hat{A}$ aymptotically stable. From (4.23) and (4.24), $P_{11} > 0$ satisfies the ARE:

$$\hat{A}^\top P_{11} + P_{11}\hat{A} - \hat{C}^\top \Sigma \hat{C} + (P_{11}\hat{B} - \hat{C}^\top \Sigma \hat{D})(\hat{D}^\top \Sigma \hat{D})^{-1}(\hat{B}^\top P_{11} - \hat{D}^\top \Sigma \hat{C}) = 0. \tag{4.26}$$

Since $\hat{A}$ is asymptotically stable, it follows from (4.24) that the ARE (4.26) has a stabilizing solution, and that this solution is equal to $P_{11} > 0$. Because of this, the Hamiltonian matrix associated with the ARE (4.26) has no eigenvalues on the imaginary axis. Consequently, (4.26) also has an anti-stabilizing solution, say $\hat{P}_+$. Since $\hat{P}_+ > P_{11} > 0$, by Proposition 4.4.2, $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$.

5.) Clearly, $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$. The input cardinality of $\mathfrak{B}$ is equal to the column dimension of $D = \hat{D}$, which is equal to the input cardinality of $\hat{\mathfrak{B}}$. $\square$

4.6.1 Interpretation in terms of amount of dissipated supply

Convincing physical interpretations of balanced truncation for passive and bounded real systems have been given before, see for example [48] and [52]. Basically, the idea is that those states are neglected that require a relatively large amount of supply to be reached in the past, but contribute little to the supply that can be extracted in the future. In this subsection we will give an alternative physical interpretation of Σ-balanced reduction in terms of *dissipation of supply.*

Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{q}}_{\mathrm{cont}}$ be strictly Σ-dissipative on $\mathbb{R}_-$. Then it is also strictly Σ-dissipative, and hence for some $\epsilon > 0$ we have

$$\int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt \geq \epsilon \int_{-\infty}^{\infty} \|w(t)\|^2 dt$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}$. The left hand side of this inequality is equal to the total supply that is dissipated if the system $\mathfrak{B}$ is taken through the trajectory w. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV-representation of $\mathfrak{B}$. As before, for a given $x_0 \in \mathbb{R}^{\mathrm{n}}$, the state space, we denote by $\mathfrak{B}(x_0)$ the subset of $\mathfrak{B}$ of all trajectories $w \in \mathfrak{B}$ such that the (unique) corresponding state trajectory x satisfies $x(0) = x_0$. It is well-known that the total supply that is dissipated depends on the gap $K_+ - K_-$ between the extremal storage functions. This is made precise as follows:

Proposition 4.6.2. *Let $x_0 \in \mathbb{R}^{\mathrm{n}}$. Then we have*

$$\inf_{w \in \mathfrak{B}(x_0) \cap \mathfrak{L}_2(\mathbb{R})} \int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt = x_0^\top (K_+ - K_-) x_0.$$

Proof. Let $x_0 \in \mathbb{R}^{\mathrm{n}}$ and $w \in \mathfrak{B}(x_0) \cap \mathfrak{L}_2(\mathbb{R})$. Denote $w_- := w|_{\mathbb{R}_-}$ and $w_+ := w|_{\mathbb{R}_+}$. Recall that $w \in \mathfrak{B}(x_0)$ if and only if $R_-(w_-) = x_0$ and $R_+(w_+) = x_0$. Thus we have:

$$\begin{aligned}\int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt &= \int_{-\infty}^{0} w(t)^\top \Sigma w(t) dt - (-\int_{0}^{\infty} w(t)^\top \Sigma w(t) dt) \\ &\geq V_{req}(w_+) - V_{av}(w_-) = x_0^\top K_+ x_0 - x_0^\top K_- x_0.\end{aligned}$$

Now let $\epsilon > 0$. There exists $w_1 \in \mathfrak{B}(x_0)$ such that $\int_{-\infty}^{0} w_1(t)^\top \Sigma w_1(t) dt \leq x_0^\top K_+ x_0 + \epsilon/2$, and $w_2 \in \mathfrak{B}(x_0)$ such that $-\int_{0}^{\infty} w_2(t)^\top \Sigma w_2(t) dt \geq x_0^\top K_- x_0 - \epsilon/2$. For the concatenation w of w_1 and w_2 at $t = 0$ (which is again in $\mathfrak{B}(x_0)$) we then have $\int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt \leq x_0^\top K_+ x_0 - x_0^\top K_- x_0 + \epsilon$. This proves the claim of the proposition. $\square$

Now let $1 > \sigma_1 \geq \sigma_2 \geq \ldots \sigma_{\mathtt{n}} > 0$ be the Σ-characteristic values of $\mathfrak{B}$, and let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV-representation of $\mathfrak{B}$, this time Σ-balanced. Then we have $K_- = \Pi$ and $K_+ = \Pi^{-1}$. Thus in Σ-balanced cordinates, the gap is given by $\Pi^{-1} - \Pi = \mathrm{diag}(\frac{1}{\sigma_1} - \sigma_1, \frac{1}{\sigma_2} - \sigma_2, \ldots, \frac{1}{\sigma_{\mathtt{n}}} - \sigma_{\mathtt{n}})$. Therefore, in Σ-balanced coordinates $x_0 = (\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}})$ we have

$$\inf_{w \in \mathfrak{B}(x_0) \cap \mathfrak{L}_2(\mathbb{R})} \int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt = \sum_{i=1}^{\mathtt{n}} \xi_i^2 (\frac{1}{\sigma_i} - \sigma_i).$$

If $x_0 = e_i$, the ith standard basis vector of $\mathbb{R}^{\mathtt{n}}$, then for $w \in \mathfrak{B}(e_i)$ the total dissipated supply is at least equal to $\frac{1}{\sigma_i} - \sigma_i$. Note that $0 < \frac{1}{\sigma_1} - \sigma_1 \leq \frac{1}{\sigma_2} - \sigma_2 \leq \ldots \leq \frac{1}{\sigma_{\mathtt{n}}} - \sigma_{\mathtt{n}}$. Thus, a nice physical interpretation of Σ-balanced reduction is that the reduction procedure 'removes' states that correspond to trajectories along which a relatively large amount of supply is dissipated.

4.7 Error bounds

4.7.1 A one-step frequency domain inequality

Starting with a strictly Σ-dissipative system $\mathfrak{B}$ with $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-balanced, Σ-normalized minimal DV-representation. Let $G(\xi) = D + C(\xi I - A)^{-1}B$. Assume that the *distinct* Σ-characteristic values of $\mathfrak{B}$ are $\sigma_1 > \sigma_2 > \ldots > \sigma_N$, where σ_i appears $\mathtt{n}_i$ times. Then $\Pi = \mathrm{diag}(\sigma_1 I_1, \sigma_2 I_2, \ldots, \sigma_N I_N)$, with I_i the $\mathtt{n}_i \times \mathtt{n}_i$ identity matrix.

Suppose now that we do a one-step reduction: partition $\Pi = \mathrm{blockdiag}(\Pi_1, \Pi_2)$, with $\Pi_2 = \sigma_N I_N$. Let $\mathfrak{B}_{\mathrm{trunc}} = B_{DV}(A_{11}, B_1, C_1, D)_{\mathrm{ext}}$ be the truncated behavior as defined in step 2. of our algorithm. Let $G_1(\xi) = D + C_1(\xi I - A_{11})^{-1}B_1$. Let $\hat{\mathfrak{B}} = \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\mathrm{ext}}$ be the reduced order behavior, and $\hat{G}(\xi) = \hat{D} + \hat{C}(\xi I - \hat{A})^{-1}\hat{B}$. Obviously, $G_1 = \hat{G}$. We will now derive a property of the *error transfer matrix* $E := G - \hat{G}$.

Theorem 4.7.1. *The rational matrix E is stable. For all $\omega \in \mathbb{R}$ we have*

$$-E^\top(-i\omega)\Sigma E(i\omega) \leq \frac{4\sigma_N^2}{1 - \sigma_N^2} I_N. \tag{4.27}$$

Proof. Denote I_N by I. By straightforward calculation it can be shown that the difference $E = G - \hat{G} = G - G_1$ is equal to $E(\xi) = C(\xi)(sI - A(\xi))^{-1}B(\xi)$, where

$$\begin{aligned} A(\xi) &:= A_{22} + A_{21}(\xi I - A_{11})^{-1}A_{12}, \\ B(\xi) &:= B_2 + A_{21}(\xi I - A_{11})^{-1}B_1, \\ C(\xi) &:= C_2 + C_1(\xi I - A_{11})^{-1}A_{12}. \end{aligned}$$

Partition Π = blockdiag(Π_1,Π_2), with $\Pi_2 = \sigma_N I_N$. Thus, $(\Pi^{-1} - \Pi)^{-1}$ = blockdiag($(\Pi_1^{-1} - \Pi_1)^{-1}$,$(\Pi_2^{-1} - \Pi_2)^{-1}$). Since our DV-representation is Σ normalized, it follows from straightforward calculation, using Lemma 4.5.5, that

$$A(-\xi)^\top \Pi_2 + \Pi_2 A(\xi) - C(-\xi)^\top \Sigma C(\xi) = 0, \tag{4.28}$$

$$A(\xi)(\Pi_2^{-1} - \Pi_2)^{-1} + (\Pi_2^{-1} - \Pi_2)^{-1}A(-\xi)^\top + B(\xi)B(-\xi)^\top = 0. \tag{4.29}$$

Since $\Pi_2 = \sigma_N I$ and $(\Pi_2^{-1} - \Pi_2)^{-1} = \frac{\sigma_N}{1-\sigma_N^2} I$, we have

$$\begin{aligned} & \sigma_N A(-\xi)^\top + \sigma_N A(\xi) - C(-\xi)^\top \Sigma C(\xi) = 0 \\ \Leftrightarrow \quad & \sigma_N[-\xi I - A(-\xi)^\top] + \sigma_N[\xi I - A(\xi)] = -C(-\xi)^\top \Sigma C(\xi) \\ \Leftrightarrow \quad & \sigma_N[-\xi I - A(-\xi)^\top]^{-1} + \sigma_N[\xi I - A(\xi)]^{-1} \\ & = \; -[-\xi I - A(-\xi)^\top]^{-1}C(-\xi)^\top \Sigma C(\xi)[\xi I - A(\xi)]^{-1} \\ \Leftrightarrow \quad & \sigma_N B(-\xi)^\top[-\xi I - A(-\xi)^\top]^{-1}B(\xi) + \sigma_N B(-\xi)^\top[\xi I - A(\xi)]^{-1}B(\xi) \\ & = \; -B(-\xi)^\top[-\xi I - A(-\xi)^\top]^{-1}C(-\xi)^\top \; \Sigma C(\xi)[\xi s I - A(\xi)]^{-1}B(\xi) \\ \Leftrightarrow \quad & \sigma_N R(-\xi)^\top + \sigma_N R(\xi) = -E(-\xi)^\top \Sigma E(\xi), \end{aligned} \tag{4.30}$$

where $R(\xi) := B(-\xi)^\top[\xi I - A(\xi)]^{-1}B(\xi)$ and $E(\xi) = C(\xi)(\xi I - A(\xi))^{-1}B(\xi)$ as above. Similarly, from equation (4.29) we get

$$\frac{\sigma_N}{1-\sigma_N^2} R(-\xi)^\top + \frac{\sigma_N}{1-\sigma_N^2} R(\xi) = R(-\xi)^\top R(\xi). \tag{4.31}$$

Now, from (4.30) and (4.31)

$$\begin{aligned} -E(-\xi)^\top \Sigma E(\xi) &= \sigma_N R(-\xi)^\top + \sigma_N R(\xi) \\ &= 2\sigma_N R(-\xi)^\top + 2\sigma_N R(\xi) - (1-\sigma_N^2)R(-\xi)^\top R(\xi) \\ &= \frac{4\sigma_N^2}{1-\sigma_N^2} I - \left[\sqrt{1-\sigma_N^2}R(-\xi)^\top - \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}} I\right] \\ & \qquad \left[\sqrt{1-\sigma_N^2}R(\xi) - \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}} I\right]. \end{aligned}$$

Now put $\xi = i\omega$ to obtain

$$
\begin{aligned}
-E(-i\omega)^\top \Sigma E(i\omega) &= \frac{4\sigma_N^2}{1-\sigma_N^2} I - \left[\sqrt{1-\sigma_N^2} R(-i\omega) - \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}} I\right]^\top \\
&\qquad \left[\sqrt{1-\sigma_N^2} R(i\omega) - \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}} I\right] \\
&\leq \frac{4\sigma_N^2}{1-\sigma_N^2} I.
\end{aligned}
$$

□

Remark 4.7.2. A system $\mathfrak{B}$ with minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$, with $G(\xi) = D + C(\xi I - A)^{-1}B$, is Σ-dissipative if and only if $G^\top(-i\omega)\Sigma G(i\omega) \geq 0$ for all ω (see [66]). By Theorem 4.7.1 for the transfer matrix E of the error system $\mathfrak{B}_{err}$ we have $E^\top(-i\omega)\Sigma E(i\omega) \geq -\frac{4\sigma_N^2}{1-\sigma_N^2} I_N$. Thus, if σ_N is small, then $\mathfrak{B}_{err}$ is close to being dissipative. The inequality (4.27) is equivalent to

$$
-\int_{-\infty}^{\infty} w_e^\top \Sigma w_e dt \leq \frac{4\sigma_N^2}{1-\sigma_N^2} \|v\|_2^2 \tag{4.32}
$$

for all $w_e, v \in \mathfrak{L}_2(\mathbb{R})$ such that $(w_e, v) \in \mathfrak{B}_{err}$. The left hand side is the total amount of supply that is delivered by the error system $\mathfrak{B}_{err}$ when it goes through the trajectory w_e. If σ_N is small then this amount is small (of course modulo scaling of of v). Thus, (4.32) is an inequality providing an upper bound to the total supply delivered by the non Σ-dissipative error system. In the next subsection we show that in the special case of strictly bounded real systems, this inequality leads to an actual error bound.

4.7.2 Error bounds for balanced reduction of strictly bounded real systems

In this section, we consider bounded real balancing of input/state/output systems. Consider the equations

$$
\dot{x} = \bar{A}x + \bar{B}u, \;\; y = \bar{C}x + \bar{D}u, \tag{4.33}
$$

with $(\bar{A}, \bar{B})$ controllable and $(\bar{C}, \bar{A})$ observable. These equations represent the input/output system $\mathfrak{B}$ given by

$$
\mathfrak{B} = \{(u,y) \mid \text{ there exists } x \text{ such that (4.33) holds}\}.
$$

By Definition 2.7.1, $\mathfrak{B}$ is strictly bounded real if it is strictly Σ-dissipative on $\mathbb{R}^-$, with Σ given by

$$\Sigma = \begin{pmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{pmatrix}.$$

In [48], a bounded real balanced truncation scheme was introduced and an $\mathcal{H}^\infty$ error bound was provided. Let $H(\xi) := \bar{D} + \bar{C}(\xi I - \bar{A})^{-1}\bar{B}$ be the transfer function from input u to output y of the original system and $\hat{H}(\xi)$ that of the reduced order model. In [48] the following error bound formula was derived: $\|H - \hat{H}\|_\infty \leq 2\sum_{i=k+1}^{N} \sigma_i$, where $\sigma_i, i = 1 \dots N$ are the distinct Σ-characteristic values of $\mathfrak{B}$. We now use the theory of Σ-balancing developed in this chapter to investigate this model reduction procedure from a different angle. Note that $\mathfrak{B}$ also has a minimal DV-representation

$$\begin{aligned} \dot{x} &= \bar{A}x + \bar{B}v, \\ \begin{pmatrix} u \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ \bar{C} \end{pmatrix} x + \begin{pmatrix} I \\ \bar{D} \end{pmatrix} v. \end{aligned}$$

In order to apply the theory developed in this chapter, we should first transform the above DV-representation into a Σ-normalized DV-representation for $\mathfrak{B}$ and, next, put it into Σ-balanced form. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-balanced Σ-normalized DV-representation of $\mathfrak{B}$. Apply then the algorithm outlined in section 4.6 to obtain a reduced order behavior $\hat{\mathfrak{B}}$ which is again strictly bounded real (by Theorem 4.6.1). The DV-representation $\mathfrak{B}_{DV}(\hat{A},\hat{B},\hat{C},\hat{D})$ of $\hat{\mathfrak{B}}$ is again Σ-normalized. Next, we investigate the frequency domain inequality (4.27) for this special case. Let $G(\xi) := C(\xi I - A)^{-1}B + D$ and $\hat{G}(\xi) = \hat{C}(\xi I - \hat{A})^{-1}\hat{B} + \hat{D}$. Partition

$$G = \begin{pmatrix} G_u \\ G_y \end{pmatrix},$$

compatible with the partition $w = (u,y)$ (i.e., $u = G_u v$ and $y = G_y v$). Likewise, define $\hat{G}_u$ and $\hat{G}_y$. Assume now that we truncate one step, as explained in subsection 4.7.1. According to Theorem 4.7.1, for all $\omega \in \mathbb{R}$ we have

$$\begin{aligned} & [G_y(-i\omega) - \hat{G}_y(-i\omega)]^\top [G_y(i\omega) - \hat{G}_y(i\omega)] \\ \leq \;& \frac{4\sigma_N^2}{1-\sigma_N^2} I + [G_u(-i\omega) - \hat{G}_u(-i\omega)]^\top [G_u(i\omega) - \hat{G}_u(i\omega)]. \end{aligned}$$

This implies that if we 'drive' both $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ with the same driving variable trajectory $v \in \mathfrak{L}_2(\mathbb{R})$ with $\|v\|_2 = 1$, with corresponding input trajectories $u,\hat{u} \in \mathfrak{L}_2(\mathbb{R})$ and $y,\hat{y} \in \mathfrak{L}_2(\mathbb{R})$ for $\mathfrak{B}$ and $\hat{\mathfrak{B}}$, respectively, then we have

$$\|y - \hat{y}\|_2^2 \leq \frac{4\sigma_N^2}{1 - \sigma_N^2} + \|u - \hat{u}\|_2^2,$$

Thus, if σ_N is small and u is close to $\hat{u}$, then the corresponding outputs y and $\hat{y}$ are close in $\mathfrak{L}_2$-norm. This formula is valid for one-step truncation only. If we truncate according to $\sigma_1 > \sigma_2 > \ldots \sigma_{\mathtt{k}} > \sigma_{\mathtt{k+1}} > \ldots > \sigma_N$ at the $(\mathtt{k} + \mathtt{1})$st characteristic value, then using the triangular inequality we get

$$\|y - \hat{y}\|_2 \leq 2\sqrt{\sum_{i=\mathtt{k+1}}^{N} \frac{\sigma_i^2}{1 - \sigma_i^2}}.$$

Here, the driving variable v should be such that the inputs of all ith order truncations ($i = 0,1,2,\ldots,\mathtt{k}$) are the same.